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Adaptive Neural Control of MIMO Nonlinear Systems with a Block-Triangular Pure-Feedback Control Structure

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***Abstract*—This paper presents adaptive neural tracking control for a class of uncertain multi-input-multi-output (MIMO) nonlinear systems in block-triangular form. All subsystems within these MIMO nonlinear systems are of completely nonaffine pure-feedback form and allowed to have different orders. To deal with the nonaffine appearance of the control variables, the mean value theorem (MVT) is employed to transform the systems into a block-triangular strict-feedback form with control coefficients being couplings among various inputs and outputs. A systematic procedure is proposed for the design of a new singularity-free adaptive neural tracking control strategy. Such a design procedure can remove the couplings among subsystems and hence avoids the possible circular control construction problem. As a consequence, all the signals in the closed-loop system are guaranteed to be semiglobally uniformly ultimately bounded (SGUUB). Moreover, the outputs of the systems are ensured to converge to a small neighborhood of the desired trajectories. Simulation studies verify the theoretical findings revealed in this work.**

***Index Terms*—Adaptive neural control, backstepping, neural networks (NN), coupling, multi-input-multi-output (MIMO) nonlinear systems.**

I. INTRODUCTION

Neural networks (NN) can approximate continuous functions to any desired accuracy by learning and parallel processing [1]. Due to such a property, a lot of effort has been invested on adaptive NN control for single-input-single-output (SISO) nonlinear systems in recent years (see [2]-[4] and references therein). For multi-input-multi-output (MIMO) nonlinear systems, where couplings, usually with uncertainties, exist among various inputs and outputs, the control problem becomes much more complex and attracts a growing number of research interest [5]-[10]. For example, in [5]-[8], adaptive control was proposed for MIMO nonlinear systems with parametric uncertainties in the input coupling matrix. To decouple the couplings among system inputs, these methods require the estimate of the "decoupling matrix" to be invertible during parameter adaptation period. The possible singularity problem thus has to be handled when inverting the estimated decoupling matrix. To avoid the difficulty of dealing with low-rank decoupling matrices, some researchers adopted different methodologies. In [9], an integral Lyapunov-based adaptive NN controller was developed for MIMO nonlinear systems with nonparametric uncertainties in both the input coupling matrix and the last equation of each subsystem within system interconnections. Since this method does not try to cancel the decoupling matrix when linearizing the system, the necessity of matrix inversion vanishes and the singularity problem is thus removed. In the follow-up work [10], the authors further considered the control problem of MIMO block-triangular strict-feedback nonlinear systems. Within these systems, the plants to be controlled contain couplings with unknown nonlinearities and/or parametric uncertainties. Besides the coupling terms in the input matrices, system interconnections are allowed in every equation of each subsystem, rather than only in the last equation. By exploring the special structure of the MIMO nonlinear systems, the adaptive NN control developed in [10] avoids the singularity problem completely without using projection algorithms [5].

It is noteworthy that the aforementioned adaptive NN control is applicable only for MIMO affine nonlinear systems. To control MIMO nonaffine nonlinear systems containing nonaffine appearances, it is much more difficult to find the explicit virtual control and actual control to stabilize the systems under study. Moreover, when the desired virtual control and actual control are approximated using NN in the backstepping design, as carried out in [9], [10], the actual control will be generally involved as the input of the NN approximation, whereas the NN approximation is a part of the actual control. As mentioned in [2], the extension of controls designed for affine systems [9], [10] to nonaffine systems will lead to a circular construction of the actual control. The problem of circular construction in controlling SISO nonaffine pure-feedback systems has been solved in [2], [11], [12], [13], [14]. In [2], [11], the main idea is to refrain from constructing an overall Lyapunov function for the closed-loop system, which can be realized by integrating the backstepping method, input-to-state stability analysis and the small gain theorem in the control system design. In [12], [13], a filtered signal was introduced to circumvent the potential circular control problem since most actuators have low-pass properties. In the follow-up work [14], by introducing a set of alternative state variables and the corresponding transformation, state-feedback

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control of the pure-feedback system can be viewed as output-feedback control of a canonical system. And consequently, the previously encountered circular control problem was also circumvented. Unfortunately, it is nontrivial to extend the design method of SISO to MIMO nonlinear systems due to various couplings involved. Therefore, it remains an open problem to establish an effective design procedure that can simultaneously deal with couplings and the possible circular control construction problem in MIMO nonaffine nonlinear systems.

Motivated by the aforementioned problems, in this paper we consider the design procedure for a class of MIMO nonlinear continuous-time systems which are more general than those studied in [10]. Specifically, these systems possess a block-triangular control structure, with each subsystem being of the completely nonaffine pure-feedback form, and couplings in the forms of unknown nonlinearities in every equation of each subsystem. By using the mean value theorem (MVT), the MIMO block-triangular pure-feedback systems are firstly transformed into a MIMO block-triangular strict-feedback form similar to that considered in [10], whereas the control coefficients are allowed to be nonaffine rather than affine appearances required in [10]. Based on the transformed systems, a systematic design procedure is then developed for the design of a new singularity-free adaptive neural control. All the signals in the closed-loop system are guaranteed to be semiglobally uniformly ultimately bounded (SGUUB) and the outputs of the systems are proven to converge to a small neighborhood of the desired trajectories. The control performance of the closed-loop system is guaranteed by suitably choosing design parameters. Simulation results are finally presented to demonstrate the effectiveness of the proposed control.

The main contributions of this paper are as follows.

(i) To the best of our knowledge, it is the first time, in the literature, that the tracking control problem of block-triangular MIMO nonlinear continuous-time systems with each subsystem having the completely nonaffine pure-feedback form is investigated.

(ii) A systematic procedure is developed for the design of an adaptive NN control such that, for the derivatives of Lyapunov function candidates with respect to the control variables, the affine parts can be guaranteed to be stabilized and the nonaffine parts, which are couplings of system inputs and outputs among subsystems, can be guaranteed to be non-positive. Due to their negative semi-definiteness, the nonaffine parts can be removed in the derivatives of Lyapunov function candidates, which simplifies the control design process and provides the following advantages: the couplings among various inputs and outputs have been completely removed without estimating the "decoupling matrix" as carried out in [5]-[8], and subsequently, the aforementioned circular control construction problem has been avoided.

(iii) Despite the interconnections between the subsystems, the stability of the whole closed-loop system can be established by analyzing individual subsystems separately, much simpler than the analysis based on a complex nested iterative manner in [10].

The rest of the paper is organized as follows. Section II presents the problem formulation and preliminaries. In Section III we describe the proposed adaptive neural control along with the main theoretical results. Section IV provides a simulation example to illustrate the effectiveness of the proposed approach. Finally, in Section V we draw our conclusion.

Throughout the paper, $A := B$ denotes that B is defined as A, $\|\cdot\|$ denotes the Euclidean norm of vectors and induced norm of matrices, $\lambda_{max}(M)$ denotes the largest eigenvalue of a square matrix M, i, j and l denote integer indices, and i_j denotes the subscription of the i_jth component of the corresponding items in the jth subsystem.

II. Problem Statement and Preliminaries

A. Problem Statement

Consider the following MIMO nonlinear systems with each subsystem having the completely nonaffine pure-feedback form

$$\left\{\begin{array}{ll} \dot{x}_{j,i_j} & = f_{j,i_j}(\bar{x}_{1,(i_j-\varrho_{j1})}, \bar{x}_{2,(i_j-\varrho_{j2})}, \cdots, \\ & \quad \bar{x}_{m,(i_j-\varrho_{jm})}, x_{j,i_j+1}), i_j = 1,2,\cdots,\rho_j - 1 \\ \dot{x}_{j,\rho_j} & = f_{j,\rho_j}(X, \bar{u}_j, d_j(t)) \\ y_j & = x_{j,1}, \quad j = 1,2,\cdots,m \end{array}\right. \tag{1}$$

with

x_{j,i_j}	$\in R$, the i_jth state of the jth subsystem;
$\bar{x}_{j,i_j}$	$[x_{j,1}, x_{j,2}, \cdots, x_{j,i_j}]^T \in R^{i_j}$;
u_j	$\in R$, the input of the jth subsystem;
$\bar{u}_j$	$[u_1, u_2, \cdots, u_j]^T \in R^j$;
y_j	$\in R$, the output of the jth subsystem;
f_{j,i_j}	the unknown nonlinear functions;
ρ_j	the order of the jth subsystem;
ϱ_{jl}	$\rho_j - \rho_l, l = 1,2,\cdots,m$;
$d_j(t)$	$\in R$, the external disturbance;
X	$[\bar{x}_{1,\rho_1}, \bar{x}_{2,\rho_2}, \cdots, \bar{x}_{m,\rho_m}]^T \in R^{\Sigma_{k=1}^m \rho_k}$, the vector of all state variables in the complete system.

Assume that $f_{j,i_j}(\cdot)$ and $f_{j,\rho_j}(\cdot,\cdot,0), i_j = 1,2,\cdots,\rho_j-1$, $j = 1,2,\cdots,m$, are smooth functions of their arguments, with $f_{j,\rho_j}(\cdot,0,\cdot)$ satisfying the Lipschitz condition, and $d_j(t), j = 1,2,\ldots,m$, are bounded by unknown positive constants d_j^*, i.e., $|d_j(t)| \leq d_j^*$.

Remark 1: For the order differences ϱ_{jl}, as introduced in [10], there exist three cases to be considered: a) when $j = l$, then $\varrho_{jl} = 0$, and accordingly state vector $\bar{x}_{j,(i_j-\varrho_{jl})} = \bar{x}_{j,i_j}$ exists in system (1); b) when $j \neq l$ and $i_j - \varrho_{jl} \leq 0$, then the corresponding vector $\bar{x}_{l,(i_j-\varrho_{jl})}$ does not exist, and does not appear in system (1); and c) when $j \neq l$ and $i_j - \varrho_{jl} > 0$, then state vector $\bar{x}_{l,(i_j-\varrho_{jl})}$ exists in system (1).

Remark 2: Compared with MIMO nonlinear system studied in [10], where each subsystem is limited to the affine strict-feedback form, system (1) is more general in the sense that it includes not only the system inputs, $\bar{u}_j$, the control signals of the first to the jth subsystem, but also the completely nonaffine properties, which represent a large class of nonlinearities including the affine strict-feedback form. These properties imply that there exist strong couplings among the system states and inputs, and accordingly cause the difficulty in finding stable controllers for system (1). In reality, many practical systems possess these features, such as biochemical processes [1], [15], flight control systems [16], mechanical systems [17], etc. Recent examples of practical systems falling into this category are dynamic models for a small-scale autonomous helicopter [18].

The control objective is to synthesize an adaptive neural tracking control for system (1) such that all the signals in the closed-loop system remain SGUUB, while the output y_j tracks the reference signal $y_{rj} \in R$, the output of the following reference model

$$\begin{aligned} \dot{x}_{ri} &= f_{ri}(x_r), 1 \le i \le n \\ y_{rj} &= x_{rj}, 1 \le j \le m \le n \end{aligned} \tag{2}$$

where $x_r = [x_{r1}, x_{r2}, \cdots, x_{rm}]^T \in R^m$ is the measured state, and $f_{ri}, i = 1, 2, \cdots, n$ are known smooth nonlinear functions. Note that the states x_r are assumed bounded in (2), i.e., $x_r \in \Omega_{x_r}, \forall t \ge 0$, where $\Omega_{x_r} \subset R^m$ is a compact set.

B. Transformation of the system representation

For the control of system (1), define

$$g_{j,i_j}(\cdot) := \frac{\partial f_{j,i_j}(\cdot)}{\partial x_{j,i_j+1}}, i_j = 1, 2, \ldots, \rho_j - 1 \tag{3}$$

$$g_{j,\rho_j}(\cdot) := \frac{\partial f_{j,\rho_j}(\cdot)}{\partial u_j}, j = 1, 2, \cdots, m. \tag{4}$$

For convenience, denote $x_{j,\rho_j+1} = u_j$ and

$$\Xi_{j,i_j} = [\bar{x}_{1,(i_j-\varrho_{j1})}, \bar{x}_{2,(i_j-\varrho_{j2})}, \cdots, \bar{x}_{m,(i_j-\varrho_{jm})}]^T.$$

Using the MVT [19], it follows that

$$\begin{aligned} f_{j,i_j}(\Xi_{j,i_j}, x_{j,i_j+1}) = h_{j,i_j}(\Xi_{j,i_j}) \\ + g_{j,i_j}(\Xi_{j,i_j}, x^c_{j,i_j+1})x_{j,i_j+1} \end{aligned} \tag{5}$$

$$\begin{aligned} f_{j,\rho_j}(X, \bar{u}_j, d_j(t)) = h_{j,\rho_j}(X, \bar{u}_{j-1}) + g_{j,\rho_j}(X, \bar{u}_{j-1}, u^c_j)u_j \\ + \delta_j(t) \end{aligned} \tag{6}$$

where

$$\begin{aligned} h_{j,i_j}(\Xi_{j,i_j}) &= f_{j,i_j}(\Xi_{j,i_j}, 0) \\ h_{j,\rho_j}(X, \bar{u}_{j-1}) &= f_{j,\rho_j}(X, \bar{u}_{j-1}, 0) \\ x^c_{j,i_j+1} &\in \left[\min\{x_{j,i_j+1}, 0\}, \max\{x_{j,i_j+1}, 0\}\right] \\ u^c_j &\in [\min\{u_j, 0\}, \max\{u_j, 0\}] \\ \delta_j(t) &= f_{j,\rho_j}(X, \bar{u}_j, d_j(t)) - f_{j,\rho_j}(X, \bar{u}_j, 0). \end{aligned}$$

Consider $f_{j,i_j}(\cdot)$ and $f_{j,\rho_j}(\cdot, \cdot, 0), i_j = 1, 2, \cdots, \rho_j - 1$, $j = 1, 2, \cdots, m$, which are unknown smooth nonlinear functions of their arguments. Accordingly, functions $h_{j,i_j}(\cdot)$ and $g_{j,i_j}(\cdot), i_j = 1, 2, \cdots, \rho_j, j = 1, 2, \cdots, m$, are unknown and smooth as well. Moreover, since $f_{j,\rho_j}(\cdot, 0, \cdot)$ satisfies the Lipschitz condition, there exists a constant $\bar{d}^*_j > 0$ such that $|\delta_j(t)| \le \bar{d}^*_j$.

From (5) and (6), system (1) can be rewritten as

$$\begin{cases} \dot{x}_{j,i_j} &= h_{j,i_j}(\Xi_{j,i_j}) + g_{j,i_j}(\Xi_{j,i_j}, x^c_{j,i_j+1})x_{j,i_j+1}, \\ & \qquad i_j = 1, 2, \cdots, \rho_j - 1 \\ \dot{x}_{j,\rho_j} &= h_{j,\rho_j}(X, \bar{u}_{j-1}) + g_{j,\rho_j}(X, \bar{u}_{j-1}, u^c_j)u_j + \delta_j(t) \\ y_j &= x_{j,1}, \quad j = 1, 2, \cdots, m. \end{cases} \tag{7}$$

Note that since $g_{j,i_j}(\cdot), i_j = 1, 2, \ldots, \rho_j, j = 1, 2, \cdots, m$, are smooth functions, they are bounded within some compact set. As commonly done in the literature, the following assumptions are made for system (7).

Assumption 1: The signs of $g_{j,i_j}(\cdot)$ are known, and there exist positive constants $\underline{g}_{j,i_j}$ and $\overline{g}_{j,i_j}$ such that for $i_j = 1, 2, \ldots, \rho_j, j = 1, 2, \cdots, m$:

(i) $|g_{j,i_j}(\cdot)| > \underline{g}_{j,i_j}, \forall(\Xi_{j,i_j}, x_{j,i_j+1})$; and
(ii) $|g_{j,i_j}(\cdot)| \le \overline{g}_{j,i_j}, \forall(\Xi_{j,i_j}, x_{j,i_j+1}) \in \Omega_{x_{j,i_{j+1}}}$

where $\Omega_{x_{j,i_{j+1}}}$ is a compact subset of the appropriate dimension space.

Assumption 1 means that g_{j,i_j} is strictly either positive or negative definite. Without loss of generality, assume that $g_{j,i_j} > \underline{g}_{j,i_j} > 0$. It should be emphasized that the bounds $\underline{g}_{j,i_j}$ and $\overline{g}_{j,i_j}$ are not necessarily known.

Remark 3: In system (7), although functions g_{j,i_j} are similar to the affine terms in MIMO systems [10], major differences lie in that: $g_{j,\rho_j}, j = 1, 2, \cdots, m$, are fully interconnection terms, i.e., they include all the state variables X; and $g_{j,i_j}, i_j = 1, 2, \cdots, \rho_j, j = 1, 2, \cdots, m$, are functions of $x_{j,i_j+1}(x_{j,\rho_j+1} := u_j)$. Therefore, system (7) is still a nonaffine nonlinear system, and is more general than the MIMO system considered in [10].

C. Gaussian radial basis networks

Due to its great capabilities in function approximation, the following Gaussian radial basis function (RBF) NN [20] is employed to approximate a smooth function $h(Z) : R^q \to R$:

$$g_{nn}(Z) = W^T S(Z) \tag{8}$$

where $Z \in \Omega_Z \subset R^q$ is the input vector, $W \in R^l$ is the weight vector, $l > 1$ is the NN nodes number, and $S(Z) = [s_1(Z), s_2(Z), \cdots, s_l(Z)]^T \in R^l$ is the basis function vector with $s_i(Z)$ being the commonly used Gaussian functions, i.e.,

$$s_i(Z) = \exp\left[\frac{-(Z - \mu_i)^T(Z - \mu_i)}{\upsilon^2}\right], i = 1, 2, \cdots, l \tag{9}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \cdots, \mu_{iq}]^T$ is the center of the receptive field, and υ is the width of the Gaussian functions.

It has been shown that any continuous function over a compact set $Z \in \Omega_Z \subset R^q$ can be approximated to any arbitrary accuracy by using network (8), i.e.,

$$h(Z) = W^{*T} S(Z) + \varepsilon(Z), \forall Z \in \Omega_Z \tag{10}$$

where W^* denotes ideal constant weights, and $\varepsilon(Z)$ is the approximation error. For clarity, write $\varepsilon := \varepsilon(Z)$.

Assumption 2: For a given continuous function $h(Z)$ and NN approximator (8), there exist ideal constant weights W^* such that $|\varepsilon| \le \varepsilon^*, \forall Z \in \Omega_Z$ with constant $\varepsilon^* > 0$.

For the Gaussian RBF NN approximator (8) and (9), the following lemma shows that there exists an upper bound on the 2-norm of vector $S(Z)$, which is useful in stability analysis of the closed-loop system.

Lemma 1: For the Gaussian RBF NN approximator (8) and (9), there exists a constant $c_{nn} > 0$ such that

$$\|S(Z)\| \le c_{nn} \tag{11}$$

where c_{nn} is the limited value of the infinite series $\{3q(k+2)^{q-1}e^{-2\rho^2k^2/\upsilon^2}\}(k=0,1,\cdots,+\infty)$, with υ being the width of the Gaussian function, q the dimension of input Z, and $\rho := \frac{1}{2}\min_{i\neq j}\|\mu_i \neq \mu_j\|$ [2], [21].

Remark 4: The positive constant c_{nn} is independent of the NN input variables Z and the NN node number l.

Note that in the following control system design, NN approximation is only guaranteed with some compact sets. Accordingly, the stability results on the considered systems are semiglobal since there exists controller(s) with a sufficiently large number of NN nodes such that all signals in the closed-loop system remain bounded, as long as the input variables of the NN remain within some compact sets that can be made as large as desired.

III. Adaptive Neural Control

In this section, adaptive neural control for MIMO system (7) is presented based on the backstepping approach and the following coordinate transformation:

$$z_{j,1} = x_{j,1} - x_{r1} \tag{12a}$$
$$z_{j,2} = x_{j,2} - \alpha_{j,1}(\Xi_{j,1}, x_r, \overline{\hat{W}}_{j,1}) \tag{12b}$$
$$\vdots$$
$$z_{j,i_j} = x_{j,i_j} - \alpha_{j,i_j-1}(\Xi_{j,i_j-1}, x_r, \overline{\hat{W}}_{j,i_j-1}), \quad 2 \leq i_j \leq \rho_j \tag{12c}$$
$$z_{j,\rho_j} = x_{j,\rho_j} - \alpha_{j,\rho_j-1}(\Xi_{j,\rho_j-1}, x_r, \overline{\hat{W}}_{j,\rho_j-1}) \tag{12d}$$

where $\alpha_{j,i_j}(\cdot), i_j = 1,2,\cdots,\rho_j-1$, are virtual controls to be determined later, and $\overline{\hat{W}}_{j,i_j} = \left[\hat{W}_{j,1}^T, \hat{W}_{j,2}^T, \cdots, \hat{W}_{j,i_j}^T\right]^T$ with $\hat{W}_{j,i_j}$ being the estimates of the ideal constant weights W_{j,i_j}^*. The design of α_{j,i_j} and $\hat{W}_{j,i_j}$ is achieved by constructing appropriate Lyapunov functions at the recursive i_jth step. The actual control u_j appears at the ρ_jth step and the design of u_j and $\hat{W}_{j,\rho_j}$ is performed to stabilize system (7).

Step 1: Differentiating both sides of (12a) yields

$$\dot{z}_{j,1} = h_{j,1}(\Xi_{j,1}) + g_{j,1}x_{j,2} - \dot{x}_{r1} \tag{13}$$

where $\Xi_{j,1} = [\bar{x}_{1,(1-\varrho_{j1})}, \cdots, \bar{x}_{j,1}, \cdots, \bar{x}_{m,(1-\varrho_{jm})}]^T$ with $\varrho_{jl} = \varrho_j - \varrho_l, l = 1,2,\cdots,m$. As mentioned in Subsection II-A, for $1-\varrho_{jl} \leq 0$, the state $\bar{x}_{l,(1-\varrho_{jl})}$ vanishes in (14). Note that $g_{j,1}$ is a function of $x_{j,2}$. From Assumption 1, we get that $g_{j,1} > \underline{g}_{j,1} > 0$, and (13) can be rewritten as

$$\dot{z}_{j,1} = \underline{g}_{j,1}\left(\nu_{j,1} + x_{j,2} + g_{j,1}^+ x_{j,2}\right) \tag{14}$$

where $\nu_{j,1} = \frac{1}{\underline{g}_{j,1}}[h_{j,1}(\Xi_{j,1}) - \dot{x}_{r1}]$ and $g_{j,1}^+ = \frac{g_{j,1}}{\underline{g}_{j,1}} - 1 > 0$.

Constructing a Lyapunov function candidate $V_{z_{j,1}} = (1/2)z_{j,1}^2$ and differentiating it, we have

$$\dot{V}_{z_{j,1}} = z_{j,1}\dot{z}_{j,1} = \underline{g}_{j,1}z_{j,1}\left(\nu_{j,1} + x_{j,2} + g_{j,1}^+ x_{j,2}\right). \tag{15}$$

The basic idea of the control design in this work is to guarantee $V_{z_{j,1}}$ to be a Lyapunov function by setting the terms involved in (15) suitably. This can be accomplished by choosing $\alpha_{j,1}^* := x_{j,2}$ as a virtual control input such that (i) $\alpha_{j,1}^* = -c_{j,1}z_{j,1} - \nu_{j,1}$, where $c_{j,1} > 0$ is a design constant, and (ii) $g_{j,1}^+ z_{j,1}\alpha_{j,1}^* \leq 0$. After these manipulations, $V_{z_{j,1}}$ becomes a Lyapunov function, and $z_{j,1} = 0$ is thus asymptotically stable.

In (14), $\nu_{j,1}$ is an unknown smooth function of $\Xi_{j,1}$ and $\dot{x}_{r1}$. Thus, $\nu_{j,1}$ can be approximated by employing a Gaussian RBF NN $W_{j,1}^T S_{j,1}(Z_{j,1})$, where $Z_{j,1} = [\Xi_{j,1}, \dot{x}_{r1}]^T \in \Omega_{Z_{j,1}}$, i.e.,

$$\nu_{j,1} = W_{j,1}^{*T} S_{j,1}(Z_{j,1}) + \varepsilon_{j,1}, \forall Z_{j,1} \in \Omega_{Z_{j,1}} \tag{16}$$

with $W_{j,1}^*$ denoting the ideal constant weights, and $|\varepsilon_{j,1}| \leq \varepsilon_{j,1}^*$ the approximation error with constant $\varepsilon_{j,1}^* > 0$.

Choose the virtual control $\alpha_{j,1}$ as

$$\alpha_{j,1} = -c_{j,1}z_{j,1} - \varpi_{j,1}\hat{W}_{j,1}^T S_{j,1}(Z_{j,1}) \tag{17}$$

where $\hat{W}_{j,1}$ is the estimate of neural weights $W_{j,1}^*$, and

$$\varpi_{j,1} = \tanh\left(\frac{\omega_{j,1}}{\epsilon_{j,1}}\right), \; \omega_{j,1} = \hat{W}_{j,1}^T S_{j,1}(Z_{j,1})z_{j,1} \tag{18}$$

with $\epsilon_{j,1} > 0$ being a small constant.

According to (12b), (16) and (17), (14) becomes

$$\dot{z}_{j,1} = -\underline{g}_{j,1}\Big(c_{j,1}z_{j,1} + \varpi_{j,1}\hat{W}_{j,1}^T S_{j,1}(Z_{j,1}) - W_{j,1}^{*T} S_{j,1}(Z_{j,1}) - \varepsilon_{j,1} - \frac{g_{j,1}}{\underline{g}_{j,1}}z_{j,2} - g_{j,1}^+\alpha_{j,1}\Big)$$

In light of Assumption 1, the following inequality holds

$$g_{j,1}^+ z_{j,1}\alpha_{j,1} = -g_{j,1}^+\left[c_{j,1}z_{j,1}^2 + \tanh\left(\frac{\omega_{j,1}}{\epsilon_{j,1}}\right)\omega_{j,1}\right] \leq 0. \tag{19}$$

Remark 5: Note that for the control of the dynamics in (13), if $g_{j,1}$ is independent of the state $x_{j,2}$, then the most commonly used control structure is $\alpha_{j,1}^* = (-h_{j,1} + \dot{x}_{r1} + \upsilon^*)/g_{j,1}$ with υ^* being a new control; and if $g_{j,1}$ is a function of $x_{j,2}$, and $\alpha_{j,1}^*$ is unknown and is approximated by NN, then the circular control construction problem will arise since $x_{j,2}$ has to be chosen as an input of the NN approximation, which is one part of the virtual control $x_{j,2}$. On the other hand, by guaranteeing the coupling term $g_{j,1}^+ z_{j,1}\alpha_{j,1}^* \leq 0$ in (15) instead of approximating $\alpha_{j,1}^*$, such problem can be completely handled using control (17), as the coupling term is removed. Moreover, to use less neurons, $\dot{x}_{r1} \in R$ is chosen as an input to $W_{j,1}^T S_{j,1}(Z_{j,1})$ rather than $x_r \in R^m$ since f_{r1} and x_r are known, and then $\dot{x}_{r1} = f_{r1}(x_r)$ is available. Thus, the online computation load is lightened. The same ideas of constructing the adaptive NN control and choosing the inputs of NN are also used in the following design steps.

Consider a Lyapunov function candidate $V_{j,1}$ as

$$V_{j,1} = \frac{1}{2}z_{j,1}^2 + \frac{\underline{g}_{j,1}}{2}\tilde{W}_{j,1}^T\Gamma_{j,1}^{-1}\tilde{W}_{j,1} \tag{20}$$

where $\tilde{W}_{j,1} = \hat{W}_{j,1} - W_{j,1}^*$, and $\Gamma_{j,1} = \Gamma_{j,1}^T > 0$ is an adaptation gain matrix.

Using (19), the derivative of $V_{j,1}$ is

$$\begin{aligned}\dot{V}_{j,1} &= \underline{g}_{j,1} z_{j,1}\bigg(- c_{j,1} z_{j,1} - \varpi_{j,1}\hat{W}_{j,1}^T S_{j,1}(Z_{j,1}) \\ &\quad + W_{j,1}^{*T} S_{j,1}(Z_{j,1}) + \varepsilon_{j,1} + \frac{g_{j,1}}{\underline{g}_{j,1}} z_{j,2} + g_{j,1}^{+}\alpha_{j,1}\bigg) \\ &\quad + \underline{g}_{j,1}\tilde{W}_{j,1}^T \Gamma_{j,1}^{-1}\dot{\hat{W}}_{j,1} \\ &\le \underline{g}_{j,1}\bigg(- c_{j,1} z_{j,1}^2 + \frac{g_{j,1}}{\underline{g}_{j,1}} z_{j,1} z_{j,2} + z_{j,1}\varepsilon_{j,1} + \Psi_{j,1}\bigg)\end{aligned} \tag{21}$$

where $\Psi_{j,1} = W_{j,1}^{*T} S_{j,1}(Z_{j,1}) z_{j,1} - \varpi_{j,1}\omega_{j,1} + \tilde{W}_{j,1}^T \Gamma_{j,1}^{-1}\dot{\hat{W}}_{j,1}$.

Consider the facts that

$$\begin{aligned}\Psi_{j,1} &= \omega_{j,1} - \varpi_{j,1}\omega_{j,1} + \tilde{W}_{j,1}^T\Gamma_{j,1}^{-1}\dot{\hat{W}}_{j,1} - \tilde{W}_{j,1}^T S_{j,1}(Z_{j,1}) z_{j,1} \\ &= \omega_{j,1} - \tanh\left(\frac{\omega_{j,1}}{\epsilon_{j,1}}\right)\omega_{j,1} \\ &\quad + \tilde{W}_{j,1}^T\Gamma_{j,1}^{-1}\left[\dot{\hat{W}}_{j,1} - \Gamma_{j,1} S_{j,1}(Z_{j,1}) z_{j,1}\right]\end{aligned} \tag{22}$$

and the following nice property of function $\tanh(\cdot)$ [22]:

$$0 \le |\omega| - \omega\tanh\left(\frac{\omega}{\epsilon}\right) \le 0.2785\epsilon, \forall\epsilon > 0, \forall\omega \in R. \tag{23}$$

Design adaptation law for $\hat{W}_{j,1}$ as

$$\dot{\hat{W}}_{j,1} = \Gamma_{j,1}\left[S_{j,1}(Z_{j,1}) z_{j,1} - \sigma_{j,1}\hat{W}_{j,1}\right] \tag{24}$$

where $\sigma_{j,1} > 0$ is a design parameter, and the σ-modification term $\sigma_{j,1}\hat{W}_{j,1}$ is designed to improve the controller robustness [23]. Without such a modification term, it will result in variation of a high-gain control since the NN weight estimates $\hat{W}_{j,1}$ might drift to very large values in the presence of the NN approximation errors [24].

Using (23) and (24), then (22) becomes

$$\Psi_{j,1} \le 0.2785\epsilon_{j,1} - \sigma_{j,1}\tilde{W}_{j,1}^T\hat{W}_{j,1}. \tag{25}$$

From Young's inequality [25], we have

$$\begin{aligned}-\sigma_{j,1}\tilde{W}_{j,1}^T\hat{W}_{j,1} &= -\sigma_{j,1}\tilde{W}_{j,1}^T\tilde{W}_{j,1} - \sigma_{j,1}\tilde{W}_{j,1}^T W_{j,1}^* \\ &\le -\sigma_{j,1}\left\|\tilde{W}_{j,1}\right\|^2 + \sigma_{j,1}\left\|\tilde{W}_{j,1}\right\|\left\|W_{j,1}^*\right\| \\ &\le -\frac{\sigma_{j,1}\left\|\tilde{W}_{j,1}\right\|^2}{2} + \frac{\sigma_{j,1}\left\|W_{j,1}^*\right\|^2}{2}\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{g_{j,1}}{\underline{g}_{j,1}} z_{j,1} z_{j,2} &\le \frac{\overline{g}_{j,1}}{\underline{g}_{j,1}}|z_{j,1}||z_{j,2}| \\ &\le \frac{c_{j,1}}{4} z_{j,1}^2 + \frac{\overline{g}_{j,1}^2}{c_{j,1}\underline{g}_{j,1}^2} z_{j,2}^2\end{aligned} \tag{27}$$

$$z_{j,1}\varepsilon_{j,1} \le \frac{c_{j,1}}{4} z_{j,1}^2 + \frac{1}{c_{j,1}}\varepsilon_{j,1}^{*2}. \tag{28}$$

Substituting (25)-(28) into (21), we have

$$\dot{V}_{j,1} \le -\frac{\underline{g}_{j,1}}{2}\left(c_{j,1} z_{j,1}^2 + \sigma_{j,1}\left\|\tilde{W}_{j,1}\right\|^2\right) + g_{j,1}^* z_{j,2}^2 + \eta_{j,1} \tag{29}$$

where $g_{j,1}^* = \frac{\overline{g}_{j,1}^2}{c_{j,1}\underline{g}_{j,1}}$ and

$$\eta_{j,1} = \underline{g}_{j,1}\left(0.2785\epsilon_{j,1} + \frac{1}{c_{j,1}}\varepsilon_{j,1}^{*2} + \frac{\sigma_{j,1}\left\|W_{j,1}^*\right\|^2}{2}\right).$$

Let $\beta_{j,1} := \min\left\{\underline{g}_{j,1} c_{j,1}, \frac{\sigma_{j,1}}{\lambda_{\max}(\Gamma_{j,1}^{-1})}\right\}$ with $\lambda_{\max}(\Gamma_{j,1}^{-1})$ being the largest eigenvalue of matrix $\Gamma_{j,1}^{-1}$, then

$$\dot{V}_{j,1} \le -\beta_{j,1} V_{j,1} + g_{j,1}^* z_{j,2}^2 + \eta_{j,1}. \tag{30}$$

Step $i_j (2 \le i_j \le \rho_j - 1)$**:** Considering (12c), its derivative is

$$\begin{aligned}\dot{z}_{j,i_j} &= [h_{j,i_j}(\Xi_{j,i_j}) + g_{j,i_j} x_{j,i_j+1}] - \dot{\alpha}_{j,i_j-1} \\ &= -g_{j,i_j-1}^* z_{j,i_j} + \underline{g}_{j,i_j}\left(\nu_{j,i_j} + x_{j,i_j+1} + g_{j,i_j}^{+} x_{j,i_j+1}\right)\end{aligned} \tag{31}$$

where $g_{j,i_j-1}^* = \frac{\overline{g}_{j,i_j-1}^2}{c_{j,i_j-1}\underline{g}_{j,i_j-1}}$, $g_{j,i_j}^{+} = \frac{g_{j,i_j}}{\underline{g}_{j,i_j}} - 1 > 0$ and

$$\nu_{j,i_j} = \frac{1}{\underline{g}_{j,i_j}}\left[h_{j,i_j}(\Xi_{j,i_j}) - \dot{\alpha}_{j,i_j-1} + g_{j,i_j-1}^* z_{j,i_j}\right] \tag{32}$$

$$\alpha_{j,i_j-1} = -c_{j,i_j-1} z_{j,i_j-1} - \varpi_{j,i_j-1}\hat{W}_{j,i_j-1}^T S_{j,i_j-1}(Z_{j,i_j-1}) \tag{33}$$

$$\varpi_{j,i_j-1} = \tanh\left(\frac{\omega_{j,i_j-1}}{\epsilon_{j,i_j-1}}\right)$$

with $\omega_{j,i_j-1} = \hat{W}_{j,i_j-1}^T S_{j,i_j-1}(Z_{j,i_j-1}) z_{j,i_j-1}$, $\hat{W}_{j,i_j-1}$ being the estimate of W_{j,i_j-1}^*, ϵ_{j,i_j-1} being a small positive constant, $\Xi_{j,i_j} = [\bar{x}_{1,(i_j-\varrho_{j1})}, \cdots, \bar{x}_{j,i_j}, \cdots, \bar{x}_{m,(i_j-\varrho_{jm})}]^T$, and $\varrho_{jl} = \varrho_j - \varrho_l, l = 1, 2, \cdots, m$. Again, as mentioned in Subsection II-A, if $i_j - \varrho_{jl} \le 0$, then the state vector $\bar{x}_{l,(i_j-\varrho_{jl})}$ does not appear in (31). Note that g_{j,i_j} is a function of x_{j,i_j+1}.

Constructing a Lyapunov function candidate $V_{z_{j,i_j}} = (1/2) z_{j,i_j}^2$, its derivative is

$$\begin{aligned}\dot{V}_{z_{j,i_j}} &= -g_{j,i_j-1}^* z_{j,i_j}^2 \\ &\quad + \underline{g}_{j,i_j} z_{j,i_j}\left(\nu_{j,i_j} + x_{j,i_j+1} + g_{j,i_j}^{+} x_{j,i_j+1}\right).\end{aligned} \tag{34}$$

If we choose $\alpha_{j,i_j}^* := x_{j,i_j+1}$ as a virtual control input such that (i) $\alpha_{j,i_j}^* = -c_{j,i_j} z_{j,i_j} - \nu_{j,i_j}$, where $c_{j,i_j} > 0$ is a design constant, and meanwhile (ii) $g_{j,i_j}^{+} z_{j,i_j}\alpha_{j,i_j}^* \le 0$, then, $V_{z_{j,i_j}}$ is a Lyapunov function, and $z_{j,i_j} = 0$ is asymptotically stable.

From (33), α_{j,i_j-1} is a function of Ξ_{j,i_j-1}, x_r and $\hat{W}_{j,1}, \cdots, \hat{W}_{j,i_j-1}$. Thus, $\dot{\alpha}_{j,i_j-1}$ can be written as

$$\dot{\alpha}_{j,i_j-1} = \sum_{l=1}^{m}\sum_{k=1}^{i_j-1-\rho_{jl}}\frac{\partial\alpha_{j,i_j-1}}{\partial x_{l,k}}(h_{l,k} + g_{l,k} x_{l,k+1}) + \zeta_{j,i_j-1}$$

where

$$\begin{aligned}\zeta_{j,i_j-1} &= \frac{\partial\alpha_{j,i_j-1}}{\partial x_r}\dot{x}_r \\ &\quad + \sum_{k=1}^{i_j-1}\frac{\partial\alpha_{j,i_j-1}}{\partial\hat{W}_{j,k}}\left[\Gamma_{j,k}\left(S_{j,k}(Z_{j,k}) z_{j,k} - \sigma_{j,k}\hat{W}_{j,k}\right)\right]\end{aligned} \tag{35}$$

is computable.

From (32), ν_{j,i_j} is an unknown smooth function of $\Xi_{j,i_j}, \dot{\alpha}_{j,i_j-1}$ and α_{j,i_j-1}. Considering (35), ν_{j,i_j} can be approximated by employing a Gaussian RBF NN $W_{j,i_j}^T S_{j,i_j}(Z_{j,i_j})$, i.e., ν_{j,i_j} can be expressed as

$$\nu_{j,i_j} = W_{j,i_j}^{*T} S_{j,i_j}(Z_{j,i_j}) + \varepsilon_{j,i_j}, \quad \forall Z_{j,i_j} \in \Omega_{Z_{j,i_j}} \tag{36}$$

where W_{j,i_j}^* denotes the ideal constant weights, $|\varepsilon_{j,i_j}| \le \varepsilon_{j,i_j}^*$ is the approximation error with constant $\varepsilon_{j,i_j}^* > 0$, and

$$Z_{j,i_j} = \left[\Xi_{j,i_j}, \left(\frac{\partial \alpha_{j,i_j-1}}{\partial \Xi_{j,i_j-1}}\right)^T, \zeta_{j,i_j-1}, \alpha_{j,i_j-1}\right]^T \in \Omega_{Z_{j,i_j}}.$$

Choose the virtual control α_{j,i_j} as

$$\alpha_{j,i_j} = -c_{j,i_j} z_{j,i_j} - \varpi_{j,i_j} \hat{W}_{j,i_j}^T S_{j,i_j}(Z_{j,i_j}) \tag{37}$$

where $\hat{W}_{j,i_j}$ is the estimate of neural weights W_{j,i_j}^*, and

$$\varpi_{j,i_j} = \tanh\left(\frac{\omega_{j,i_j}}{\epsilon_{j,i_j}}\right), \ \omega_{j,i_j} = \hat{W}_{j,i_j}^T S_{j,i_j}(Z_{j,i_j}) z_{j,i_j} \tag{38}$$

with $\epsilon_{j,i_j} > 0$ being a small constant.

From (12c), (36) and (37), (31) becomes

$$\begin{aligned}\dot{z}_{j,i_j} = -g_{j,i_j-1}^* z_{j,i_j} + \underline{g}_{j,i_j}\Big(& - c_{j,i_j} z_{j,i_j} \\ & - \varpi_{j,i_j} \hat{W}_{j,i_j}^T S_{j,i_j}(Z_{j,i_j}) + W_{j,i_j}^{*T} S_{j,i_j}(Z_{j,i_j}) + \varepsilon_{j,i_j} \\ & + \frac{g_{j,i_j}}{\underline{g}_{j,i_j}} z_{j,i_j+1} + g_{j,i_j}^+ \alpha_{j,i_j}\Big).\end{aligned} \tag{39}$$

According to Assumption 1, the following inequality holds

$$g_{j,i_j}^+ z_{j,i_j} \alpha_{j,i_j} = -g_{j,i_j}^+ \left[c_{j,i_j} z_{j,i_j}^2 + \tanh\left(\frac{\omega_{j,i_j}}{\epsilon_{j,i_j}}\right)\omega_{j,i_j}\right] \le 0. \tag{40}$$

Consider a Lyapunov function candidate V_{j,i_j} as

$$V_{j,i_j} = V_{j,i_j-1} + \frac{1}{2} z_{j,i_j}^2 + \frac{\underline{g}_{j,i_j}}{2} \tilde{W}_{j,i_j}^T \Gamma_{j,i_j}^{-1} \tilde{W}_{j,i_j} \tag{41}$$

where $\tilde{W}_{j,i_j} = \hat{W}_{j,i_j} - W_{j,i_j}^*$, and $\Gamma_{j,i_j} = \Gamma_{j,i_j}^T > 0$ is an adaptation gain matrix.

Using (39) and (40), the derivative of V_{j,i_j} is

$$\begin{aligned}\dot{V}_{j,i_j} \le \dot{V}_{j,i_j-1} - g_{j,i_j-1}^* z_{j,i_j}^2 + \underline{g}_{j,i_j}\Big(& - c_{j,i_j} z_{j,i_j}^2 \\ & + \frac{g_{j,i_j}}{\underline{g}_{j,i_j}} z_{j,i_j} z_{j,i_j+1} + z_{j,i_j}\varepsilon_{j,i_j} + \Psi_{j,i_j}\Big)\end{aligned} \tag{42}$$

where

$\Psi_{j,i_j} = W_{j,i_j}^{*T} S_{j,i_j}(Z_{j,i_j}) z_{j,i_j} - \varpi_{j,i_j}\omega_{j,i_j} + \tilde{W}_{j,i_j}^T \Gamma_{j,i_j}^{-1} \dot{\hat{W}}_{j,i_j}$. Consider the fact that

$$\begin{aligned}\Psi_{j,i_j} = \omega_{j,i_j} - \tanh\left(\frac{\omega_{j,i_j}}{\epsilon_{j,i_j}}\right)\omega_{j,i_j} \\ + \tilde{W}_{j,i_j}^T \Gamma_{j,i_j}^{-1}\left[\dot{\hat{W}}_{j,i_j} - \Gamma_{j,i_j} S_{j,i_j}(Z_{j,i_j}) z_{j,i_j}\right].\end{aligned} \tag{43}$$

Design adaptation law for $\hat{W}_{j,i_j}$ as

$$\dot{\hat{W}}_{j,i_j} = \Gamma_{j,i_j}\left[S_{j,i_j}(Z_{j,i_j}) z_{j,i_j} - \sigma_{j,i_j}\hat{W}_{j,i_j}\right] \tag{44}$$

where $\sigma_{j,i_j} > 0$ is a design parameter.

Using the property of function $\tanh(\cdot)$ in (23) and combining (44), then (43) becomes

$$\Psi_{j,i_j} \le 0.2785\epsilon_{j,i_j} - \sigma_{j,i_j}\tilde{W}_{j,i_j}^T \hat{W}_{j,i_j}. \tag{45}$$

Using Young's inequality [25], we have

$$-\sigma_{j,i_j}\tilde{W}_{j,i_j}^T \hat{W}_{j,i_j} \le -\frac{\sigma_{j,i_j}\left\|\tilde{W}_{j,i_j}^T\right\|^2}{2} + \frac{\sigma_{j,i_j}\left\|W_{j,i_j}^*\right\|^2}{2} \tag{46}$$

$$\frac{g_{j,i_j}}{\underline{g}_{j,i_j}} z_{j,i_j} z_{j,i_j+1} \le \frac{c_{j,i_j}}{4} z_{j,i_j}^2 + \frac{\overline{g}_{j,i_j}^2}{c_{j,i_j}\underline{g}_{j,i_j}^2} z_{j,i_j+1}^2 \tag{47}$$

$$z_{j,i_j}\varepsilon_{j,i_j} \le \frac{c_{j,i_j}}{4} z_{j,i_j}^2 + \frac{1}{c_{j,i_j}}\varepsilon_{j,i_j}^{*2} \tag{48}$$

Substituting (45)-(48) into (42), we have

$$\begin{aligned}\dot{V}_{j,i_j} \le -\beta_{j,i_j-1} V_{j,i_j-1} - \frac{\underline{g}_{j,i_j}}{2}\left(c_{j,i_j} z_{j,i_j}^2 + \sigma_{j,i_j}\left\|\tilde{W}_{j,i_j}^T\right\|^2\right) \\ + g_{j,i_j}^* z_{j,i_j+1}^2 + \eta_{j,i_j}\end{aligned} \tag{49}$$

where

$\eta_{j,i_j} = \underline{g}_{j,i_j}\left(0.2785\epsilon_{j,i_j} + \frac{1}{c_{j,i_j}}\varepsilon_{j,i_j}^{*2} + \frac{\sigma_{j,i_j}\left\|W_{j,i_j}^*\right\|^2}{2}\right) + \eta_{j,i_j-1}$.

Let $\beta_{j,i_j} := \min\left\{\beta_{j,i_j-1}, \underline{g}_{j,i_j} c_{j,i_j}, \frac{\sigma_{j,i_j}}{\lambda_{\max}\left(\Gamma_{j,i_j}^{-1}\right)}\right\}$ with $\lambda_{\max}(\Gamma_{j,i_j}^{-1})$ being the largest eigenvalue of matrix Γ_{j,i_j}^{-1}, then

$$\dot{V}_{j,i_j} \le -\beta_{j,i_j} V_{j,i_j} + g_{j,i_j}^* z_{j,i_j+1}^2 + \eta_{j,i_j}. \tag{50}$$

Step ρ_j: Considering (12d), its derivative is

$$\begin{aligned}\dot{z}_{j,\rho_j} &= \left[h_{j,\rho_j}(X, \bar{u}_{j-1}) + g_{j,\rho_j} u_j + \delta_j(t)\right] - \dot{\alpha}_{j,\rho_j-1} \\ &= -g_{j,\rho_j-1}^* z_{j,\rho_j} + \underline{g}_{j,\rho_j}\left[\nu_{j,\rho_j} + u_j + g_{j,\rho_j}^+ u_j\right] + \delta_j(t)\end{aligned} \tag{51}$$

where $g_{j,\rho_j-1}^* = \frac{\overline{g}_{j,\rho_j-1}^2}{c_{j,\rho_j-1}\underline{g}_{j,\rho_j-1}}$, $g_{j,\rho_j}^+ = \frac{g_{j,\rho_j}}{\underline{g}_{j,\rho_j}} - 1 > 0$ and

$$\nu_{j,\rho_j} = \frac{1}{\underline{g}_{j,\rho_j}}\left[h_{j,\rho_j}(X, \bar{u}_{j-1}) - \dot{\alpha}_{j,\rho_j-1} + g_{j,\rho_j-1}^* z_{j,\rho_j}\right].$$

It should be noticed that g_{j,ρ_j} is a function of u_j.

Constructing a Lyapunov function candidate $V_{z_{j,\rho_j}} = (1/2) z_{j,\rho_j}^2$, its derivative is

$$\begin{aligned}\dot{V}_{z_{j,\rho_j}} &= -g_{j,\rho_j-1}^* z_{j,\rho_j}^2 \\ &\quad + \underline{g}_{j,\rho_j} z_{j,\rho_j}\left(\nu_{j,\rho_j} + u_j + g_{j,\rho_j}^+ u_j\right) + z_{j,\rho_j}\delta_j(t).\end{aligned} \tag{52}$$

If we construct the actual control input u_j such that (i) $u_j = -c_{j,\rho_j} z_{j,\rho_j} - \nu_{j,\rho_j}$, where $c_{j,\rho_j} > 0$ is a design constant, and meanwhile (ii) $g_{j,\rho_j}^+ z_{j,\rho_j} u_j \le 0$, then, $V_{z_{j,\rho_j}}$ converges to a small neighborhood of $V_{z_{j,\rho_j}} = 0$.

From the design at the former step, it can be seen that α_{j,ρ_j-1} is a function of Ξ_{j,ρ_j-1}, x_r and $\hat{W}_{j,1}, \cdots, \hat{W}_{j,\rho_j-1}$. Thus, $\dot{\alpha}_{j,\rho_j-1}$ can be written as

$$\dot{\alpha}_{j,\rho_j-1} = \sum_{l=1}^{m}\sum_{k=1}^{\rho_l-1}\frac{\partial \alpha_{j,\rho_j-1}}{\partial x_{l,k}}(h_{l,k} + g_{l,k} x_{l,k+1}) + \zeta_{j,\rho_j-1}$$

where

$$\zeta_{j,\rho_j-1} = \frac{\partial \alpha_{j,\rho_j-1}}{\partial x_r}\dot{x}_r + \sum_{k=1}^{\rho_j-1}\frac{\partial \alpha_{j,\rho_j-1}}{\partial \hat{W}_{j,k}}\left[\Gamma_{j,k}\left(S_{j,k}(Z_{j,k})z_{j,k} - \sigma_{j,k}\hat{W}_{j,k}\right)\right] \tag{53}$$

is computable.

In (51), ν_{j,ρ_j} is an unknown smooth function of $X, \bar{u}_{j-1}, \dot{\alpha}_{j,\rho_j-1}$ and α_{j,ρ_j-1}. Considering (53), ν_{j,ρ_j} can be approximated by employing a Gaussian RBF NN $W^T_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j})$, i.e., ν_{j,ρ_j} can be written as

$$\nu_{j,\rho_j} = W^{*T}_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j}) + \varepsilon_{j,\rho_j}, \quad \forall Z_{j,\rho_j} \in \Omega_{Z_{j,\rho_j}} \tag{54}$$

where W^*_{j,ρ_j} denotes the ideal constant weights, $|\varepsilon_{j,\rho_j}| \le \varepsilon^*_{j,\rho_j}$ is the approximation error with constant $\varepsilon^*_{j,\rho_j} > 0$, and

$$Z_{j,\rho_j} = \left[X, \bar{u}_{j-1}, \left(\frac{\partial \alpha_{j,\rho_j-1}}{\partial \Xi_{j,\rho_j-1}}\right)^T, \zeta_{j,\rho_j-1}, \alpha_{j,\rho_j-1}\right]^T \in \Omega_{Z_{j,\rho_j}}.$$

Design the actual control input u_j as

$$u_j = -c_{j,\rho_j}z_{j,\rho_j} - \varpi_{j,\rho_j}\hat{W}^T_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j}) \tag{55}$$

where $\hat{W}_{j,\rho_j}$ is the estimate of neural weights W^*_{j,ρ_j}, and

$$\varpi_{j,\rho_j} = \tanh\left(\frac{\omega_{j,\rho_j}}{\epsilon_{j,\rho_j}}\right), \ \omega_{j,\rho_j} = \hat{W}^T_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j})z_{j,\rho_j} \tag{56}$$

with $\epsilon_{j,\rho_j} > 0$ being a small constant.

Then, the dynamics of z_{j,ρ_j} are governed by

$$\begin{aligned}\dot{z}_{j,\rho_j} =& -g^*_{j,\rho_j-1}z_{j,\rho_j} \\ &+ \underline{g}_{j,\rho_j}\Big(-c_{j,\rho_j}z_{j,\rho_j} - \varpi_{j,\rho_j}\hat{W}^T_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j}) \\ &+ W^{*T}_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j}) + \varepsilon_{j,\rho_j} + g^+_{j,\rho_j}u_j\Big) + \delta_j(t).\end{aligned} \tag{57}$$

According to Assumption 1, the following inequality holds

$$g^+_{j,\rho_j}z_{j,\rho_j}u_j = -g^+_{j,\rho_j}\left[c_{j,\rho_j}z^2_{j,\rho_j} + \tanh\left(\frac{\omega_{j,\rho_j}}{\epsilon_{j,\rho_j}}\right)\omega_{j,\rho_j}\right] \le 0. \tag{58}$$

Consider a Lyapunov function candidate V_{j,ρ_j} as

$$V_{j,\rho_j} = V_{j,\rho_j-1} + \frac{1}{2}z^2_{j,\rho_j} + \frac{\underline{g}_{j,\rho_j}}{2}\tilde{W}^T_{j,\rho_j}\Gamma^{-1}_{j,\rho_j}\tilde{W}_{j,\rho_j} \tag{59}$$

where $\tilde{W}_{j,\rho_j} = \hat{W}_{j,\rho_j} - W^*_{j,\rho_j}$, and $\Gamma_{j,\rho_j} = \Gamma^T_{j,\rho_j} > 0$ is an adaptation gain matrix.

Using (57) and (58), the derivative of V_{j,ρ_j} is

$$\begin{aligned}\dot{V}_{j,\rho_j} \le& \dot{V}_{j,\rho_j-1} - g^*_{j,\rho_j-1}z^2_{j,\rho_j} \\ &+ \underline{g}_{j,\rho_j}\left(-c_{j,\rho_j}z^2_{j,\rho_j} + z_{j,\rho_j}\varepsilon_{j,\rho_j} + \Psi_{j,\rho_j}\right) \\ &+ z_{j,\rho_j}\delta_j(t)\end{aligned} \tag{60}$$

where $\Psi_{j,\rho_j} = W^{*T}_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j})z_{j,\rho_j} - \varpi_{j,\rho_j}\omega_{j,\rho_j} + \tilde{W}^T_{j,\rho_j}\Gamma^{-1}_{j,\rho_j}\dot{\hat{W}}_{j,\rho_j}$.

Consider the fact that

$$\begin{aligned}\Psi_{j,\rho_j} =& \omega_{j,\rho_j} - \tanh\left(\frac{\omega_{j,\rho_j}}{\epsilon_{j,\rho_j}}\right)\omega_{j,\rho_j} \\ &+ \tilde{W}^T_{j,\rho_j}\Gamma^{-1}_{j,\rho_j}\left[\dot{\hat{W}}_{j,\rho_j} - \Gamma_{j,\rho_j}S_{j,\rho_j}(Z_{j,\rho_j})z_{j,\rho_j}\right].\end{aligned} \tag{61}$$

Design adaptation law for $\hat{W}_{j,\rho_j}$ as

$$\dot{\hat{W}}_{j,\rho_j} = \Gamma_{j,\rho_j}\left[S_{j,\rho_j}(Z_{j,\rho_j})z_{j,\rho_j} - \sigma_{j,\rho_j}\hat{W}_{j,\rho_j}\right] \tag{62}$$

where $\sigma_{j,\rho_j} > 0$ is a design parameter.

Using the property of function $\tanh(\cdot)$ in (23) and combining (62), then (61) becomes

$$\Psi_{j,\rho_j} \le 0.2785\epsilon_{j,\rho_j} - \sigma_{j,\rho_j}\tilde{W}^T_{j,\rho_j}\hat{W}_{j,\rho_j}. \tag{63}$$

Using Young's inequality [25], we have

$$\begin{aligned}-\sigma_{j,\rho_j}\tilde{W}^T_{j,\rho_j}\hat{W}_{j,\rho_j} =& -\sigma_{j,\rho_j}\tilde{W}^T_{j,\rho_j}\tilde{W}_{j,\rho_j} - \sigma_{j,\rho_j}\tilde{W}^T_{j,\rho_j}W^*_{j,\rho_j} \\ \le& -\frac{\sigma_{j,\rho_j}\left\|\tilde{W}^T_{j,\rho_j}\right\|^2}{2} + \frac{\sigma_{j,\rho_j}\left\|W^*_{j,\rho_j}\right\|^2}{2}\end{aligned} \tag{64}$$

$$z_{j,\rho_j}\varepsilon_{j,\rho_j} \le \frac{c_{j,\rho_j}}{4}z^2_{j,\rho_j} + \frac{1}{c_{j,\rho_j}}\varepsilon^{*2}_{j,\rho_j} \tag{65}$$

$$z_{j,\rho_j}\delta_j(t) \le \frac{c_{j,\rho_j}\underline{g}_{j,\rho_j}}{4}z^2_{j,\rho_j} + \frac{1}{c_{j,\rho_j}\underline{g}_{j,\rho_j}}\bar{d}^{*2}_j \tag{66}$$

Recall that at Step $\rho_j - 1$, $\dot{V}_{j,\rho_j-1}$ can be expressed as

$$\dot{V}_{j,\rho_j-1} \le -\beta_{j,\rho_j-1}V_{j,\rho_j-1} + g^*_{j,\rho_j-1}z^2_{j,\rho_j} + \eta_{j,\rho_j-1} \tag{67}$$

Substituting (63)-(67) into (60), we have

$$\begin{aligned}\dot{V}_{j,\rho_j} \le& -\beta_{j,\rho_j-1}V_{j,\rho_j-1} \\ &- \frac{\underline{g}_{j,\rho_j}}{2}\left(c_{j,\rho_j}z^2_{j,\rho_j} + \sigma_{j,\rho_j}\left\|\tilde{W}^T_{j,\rho_j}\right\|^2\right) + \eta_{j,\rho_j}\end{aligned} \tag{68}$$

where $\eta_{j,\rho_j} = \underline{g}_{j,\rho_j}\left(0.2785\epsilon_{j,\rho_j} + \frac{1}{2c_{j,\rho_j}}\varepsilon^{*2}_{j,\rho_j} + \frac{\sigma_{j,\rho_j}\left\|W^*_{j,\rho_j}\right\|^2}{2}\right) + \eta_{j,\rho_j-1} + \frac{1}{c_{j,\rho_j}\underline{g}_{j,\rho_j}}\bar{d}^{*2}_j$.

Let $\beta_{j,\rho_j} := \min\left\{\beta_{j,\rho_j-1}, \underline{g}_{j,\rho_j}c_{j,\rho_j}, \frac{\sigma_{j,\rho_j}}{\lambda_{\max}\left(\Gamma^{-1}_{j,\rho_j}\right)}\right\}$ with $\lambda_{\max}(\Gamma^{-1}_{j,\rho_j})$ being the largest eigenvalue of matrix Γ^{-1}_{j,ρ_j}, then

$$\dot{V}_{j,\rho_j} \le -\beta_{j,\rho_j}V_{j,\rho_j} + \eta_{j,\rho_j}. \tag{69}$$

Based on the above analysis, the following theorem states the stability and control performance of the closed-loop system.

Theorem 1: Consider the closed-loop system consisting of the plant (1) satisfying Assumptions 1, the reference model (2), the controller (55) and the NN weight updating laws (24), (44) and (62). Assume that there exists a sufficiently large compact set $\Omega_{Z_{j,i_j}}, i_j = 1, 2, \cdots, \rho_j, j = 1, 2, \cdots, m$, such that $Z_{j,i_j} \in \Omega_{Z_{j,i_j}}, \forall t \ge 0$. Then, for bounded initial conditions,

(i) all signals in the closed-loop system are bounded, and for the state vector X and the neural weights $\overline{\hat{W}}_j = \left[\hat{W}^T_{j,1}, \hat{W}^T_{j,2}, \cdots, \hat{W}^T_{j,\rho_j}\right]^T, j = 1, 2, \cdots, m$, they eventually converge to the compact set

$$\Omega_s := \left\{X, \overline{\hat{W}}_1, \overline{\hat{W}}_2, \cdots, \overline{\hat{W}}_m \,\middle|\, V \le \frac{\eta}{\beta}, x_r \in \Omega_{x_r}\right\} \tag{70}$$

where $\beta = \min\{\beta_{1,\rho_1}, \beta_{2,\rho_2}, \cdots, \beta_{m,\rho_m}\}$ and $\eta = \sum_{j=1}^{m} \eta_{j,\rho_j}$ are positive constants, and

(ii) the tracking error $E = [z_{1,1}, z_{2,1}, \cdots, z_{m,1}]^T \in R^m$ converges to a small neighborhood around zero by appropriately choosing design parameters.

Proof: (i) Consider the Lyapunov function candidate

$$V = \sum_{j=1}^{m} V_{j,\rho_j}. \tag{71}$$

From (69), differentiating V yields

$$\dot{V} \le \sum_{j=1}^{m}(-\beta_{j,\rho_j} V_{j,\rho_j} + \eta_{j,\rho_j}) \le -\beta V + \eta \tag{72}$$

where $\beta = \min\{\beta_{1,\rho_1}, \beta_{2,\rho_2}, \cdots, \beta_{m,\rho_m}\}$ and $\eta = \sum_{j=1}^{m} \eta_{j,\rho_j}$ are positive constants.

Let $\chi(t) := \left(X(t), \overline{\hat{W}}_1(t), \cdots, \overline{\hat{W}}_m(t)\right)$. If the initial values $\chi(0) \in \Omega_s$, where Ω_s is defined in (70), from Theorem 2.14 in [26], signals X and $\overline{\hat{W}}_j$ stay inside Ω_s, i.e., $\chi(t) \in \Omega_s, \forall t \ge 0$; if $\chi(0) \in \Omega_s^c$, where Ω_s^c denotes the complimentary set of Ω_s, then the dynamics (72) drives X and $\overline{\hat{W}}_j$ to enter and remain inside Ω_s. In summary, all z_{j,i_j} and $\hat{W}_{j,i_j}, i_j = 1, 2, \cdots, \rho_j, j = 1, 2, \cdots, m$, are uniformly ultimately bounded for bounded initial conditions.

From (12), (17) and (37), system state variables $x_{j,i_j}, i_j = 1, 2, \cdots, \rho_j, j = 1, 2, \cdots, m$, can be expressed as

$$\begin{cases} x_{j,1} &= z_{j,1} + x_{r1} \\ x_{j,i_j} &= z_{j,i_j} - c_{j,i_j-1} z_{j,i_j-1} \\ &\quad - \varpi_{j,i_j-1} \hat{W}_{j,i_j-1}^T S_{j,i_j-1}(Z_{j,i_j-1}), \\ &\quad i_j = 2, 3, \cdots, \rho_j, j = 1, 2, \cdots, m \end{cases} \tag{73}$$

From Lemma 1, we have $\|S_{j,i_j}(Z_{j,i_j})\| \le c_{nn_{j,i_j}}$ with finite constant $c_{nn_{j,i_j}} > 0$. Moreover, from the facts that the reference signals x_r are bounded and $\varpi_{j,i_j} \in (-1, 1)$, we obtain from (73) that $x_{j,i_j}, i_j = 1, 2, \cdots, \rho_j, j = 1, 2, \cdots, m$, remain bounded. Using (55), control $u_j, j = 1, 2, \cdots, m$, are bounded as well. Therefore, all signals in the closed-loop system remain bounded.

(ii) Denote $\varsigma := \eta/\beta > 0$, then (72) satisfies

$$0 \le V(t) < \varsigma + V(0)\exp(-\beta t). \tag{74}$$

From (74), we can obtain

$$\frac{1}{2}\sum_{j=1}^{m}\sum_{i_j=1}^{\rho_j} z_{j,i_j}^2 < \varsigma + V(0)\exp(-\beta t). \tag{75}$$

Furthermore, we have

$$\sum_{j=1}^{m} z_{j,1}^2 < 2\varsigma + 2V(0)\exp(-\beta t) \tag{76}$$

which implies that, given $\gamma > \sqrt{2\varsigma}$, there exists $T > 0$ such that

$$\|E\| = \left(\sum_{j=1}^{m} z_{j,1}^2\right)^{\frac{1}{2}} < \gamma, \forall t \ge T \tag{77}$$

where γ is the size of a small residual set which depends on the NN approximation error ε_{j,i_j} and controller parameters $c_{j,i_j}, \sigma_{j,i_j}, \Gamma_{j,i_j}$. ∎

Remark 6: Based on Lemma 1 and the coordinate transformation (12), although there exist interconnections between the subsystems, the stability of the whole closed-loop system can be concluded by stability analysis of individual subsystem separately without complex analysis in a nested iterative manner as in [10].

Remark 7: From (68)-(70), it can be seen that the size of Ω_s depends on $W_{j,i_j}^*, \varepsilon_{j,i_j}^*, \bar{d}_j^*$ and all design parameters. Since there is no analytical result in the NN literature to give an explicit expression of the NN node numbers l_j, the ideal constant weights W_{j,i_j}^*, and the approximation error ε_{j,i_j}^*, we here point out the following relationships: (i) increasing l_j will help to decrease ε_{j,i_j}^*, and therefore decrease Ω_s; and (ii) increasing c_{j,i_j} and decreasing σ_{j,i_j} and ϵ_{j,i_j} might lead to smaller Ω_s. However, in practical applications, there is a certain trade-off between the choice of the design parameters and the numerical precision of the tools involved in the MIMO control design [27], [28].

Remark 8: In the above systematic design procedure, by guaranteeing the coupling terms $g_{j,i_j}^+ z_{j,i_j} \alpha_{j,i_j}^* \le 0$ in the derivatives of Lyapunov function candidates rather than approximating α_{j,i_j}^*, the coupling terms have been removed, and consequently, both the circular control construction problem and the singularity problem have been handled by the developed adaptive NN control.

IV. Simulation Studies

In this section, simulation examples are presented to illustrate the effectiveness of the proposed control approach.

Example 1: Consider the following MIMO nonlinear system with each subsystem having the completely nonaffine pure-feedback form:

$$\begin{cases} \dot{x}_{1,1} &= x_{1,1} + x_{1,2} + \frac{x_{1,2}^3}{5} \\ \dot{x}_{1,2} &= x_{1,1}x_{1,2} + x_{2,1} + u_1 + \frac{u_1^3}{7} + d_1(t) \\ \dot{x}_{2,1} &= x_{1,1}x_{1,2} + x_{2,1} + u_1 + u_2 + \frac{u_2^3}{7} + d_2(t) \\ y_j &= x_{j,1}, \qquad j = 1, 2 \end{cases} \tag{78}$$

where $d_j(t) = 0.1\cos(0.01t)\cos(x_{j,1}), j = 1, 2$. Clearly, system (78) consists of two subsystems ($\rho_1 = 2; \rho_2 = 1$), with each subsystem in the nonaffine pure-feedback form. Since $1 - \rho_{12} = 0$, the state vector $\bar{x}_{2,(1-\rho_{12})}$ dose not appear in system (78).

The control objective is to make the outputs y_1 and y_2 track the desired reference trajectories y_{r1} and y_{r2}, which are the outputs of the famous van der Pol oscillator [29]

$$\begin{cases} \dot{x}_{r1} &= x_{r2} \\ \dot{x}_{r2} &= -x_{r1} + \beta(1 - x_{r1}^2)x_{r2} \\ y_{rj} &= x_{rj}, \qquad j = 1, 2 \end{cases} \tag{79}$$

where the output trajectories of the van der Pol oscillator approach a limit cycle when $\beta > 0$.

The adaptive NN controllers and the design parameters for system (78) are chosen as follows:

$$u_j = -c_{j,2} z_{j,2} - \varpi_{j,2} \hat{W}_{j,2}^T S_{j,2}(Z_{j,2}), j = 1, 2 \tag{80}$$

where $z_{1,1} = x_{1,1} - y_{r1}, z_{1,2} = x_{1,2} - \alpha_{1,1}, z_{2,1} = x_{2,1} - y_{r2}, Z_{1,1} = [x_{1,1}, \dot{x}_{r1}]^T \in R^2$ and

$$Z_{1,2} = [x_{1,1}, x_{1,2}, x_{2,1}, \frac{\partial \alpha_{1,1}}{\partial x_{1,1}}, \zeta_{1,1}, \alpha_{1,1}]^T \in R^6$$
$$Z_{2,1} = [x_{1,1}, x_{1,2}, x_{2,1}, u_1, \dot{x}_{r2}]^T \in R^5$$

with

$$\alpha_{1,1} = -c_{1,1} z_{1,1} - \varpi_{1,1} \hat{W}_{1,1}^T S_{1,1}(Z_{1,1})$$
$$\zeta_{1,1} = \frac{\partial \alpha_{1,1}}{\partial x_{r1}} \dot{x}_{r1} + \frac{\partial \alpha_{1,1}}{\partial x_{r2}} \dot{x}_{r2} + \frac{\partial \alpha_{1,1}}{\partial \hat{W}_{1,1}} \dot{\hat{W}}_{1,1}$$
$$\varpi_{1,i_1} = \tanh\left(\frac{\hat{W}_{1,i_1}^T S_{1,i_1}(Z_{1,i_1}) z_{1,i_1}}{\epsilon_{1,i_1}}\right), i_1 = 1, 2$$
$$\varpi_{2,1} = \tanh\left(\frac{\hat{W}_{2,1}^T S_{2,1}(Z_{2,1}) z_{2,1}}{\epsilon_{2,1}}\right)$$

and NN weights $\hat{W}_{j,i_j}$ are updated by

$$\dot{\hat{W}}_{j,i_j} = \Gamma_{j,i_j}[S_{j,i_j}(Z_{j,i_j}) z_{j,i_j} - \sigma_{j,i_j} \hat{W}_{j,i_j}]. \tag{81}$$

In practice, the selection of the centers and widths of RBF NN has a great influence on the performance of the designed controller. According to [10], Gaussian RBF NN arranged on a regular lattice on R^n can uniformly approximate sufficiently smooth functions on closed, bounded subsets. In the following simulation studies, NN $\hat{W}_{1,1}^T S_{1,1}(Z_{1,1})$ contains 9 nodes (i.e., $l_{1,1} = 9$), with widths $\upsilon_{1,1,k} = 2$ $(k = 1, 2, \cdots, l_{1,1})$ and centers $\mu_{1,1,k}$ $(k = 1, 2, \cdots, l_{1,1})$ evenly spaced in $[-2.5, 2.5] \times [-7, 7]$; $\hat{W}_{1,2}^T S_{1,2}(Z_{1,2})$ contains 729 nodes (i.e., $l_{1,2} = 729$), with widths $\upsilon_{1,2,k} = 4$ $(k = 1, 2, \cdots, l_{1,2})$ and centers $\mu_{1,2,k}$ $(k = 1, 2, \cdots, l_{1,2})$ evenly spaced in $[-2.5, 2.5] \times [-3, 2] \times [-2, 2] \times [-1.9, -1.6] \times [-4, 4] \times [-4, 4]$; $\hat{W}_{2,1}^T S_{2,1}(Z_{2,1})$ contains 243 nodes (i.e., $l_{2,1} = 243$), with widths $\upsilon_{2,1,k} = 2$ $(k = 1, 2, \cdots, l_{2,1})$ and centers $\mu_{2,1,k}$ $(k = 1, 2, \cdots, l_{2,1})$ evenly spaced in $[-2.5, 2.5] \times [-3, 2] \times [-2, 2] \times [-3.5, 3.5] \times [-4, 4]$.

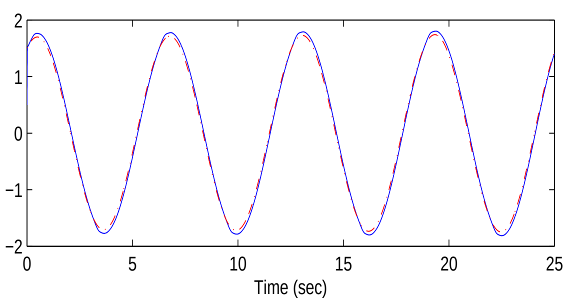

Fig. 1. Output y_1 ("—") follows y_{r1} ("- -")

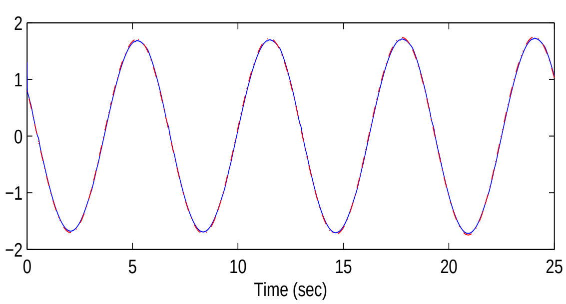

Fig. 2. Output y_2 ("—") follows y_{r2} ("- -")

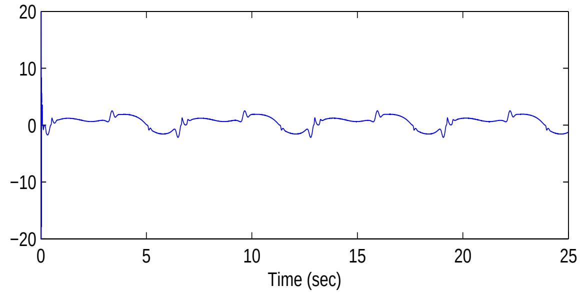

Fig. 3. Control input u_1

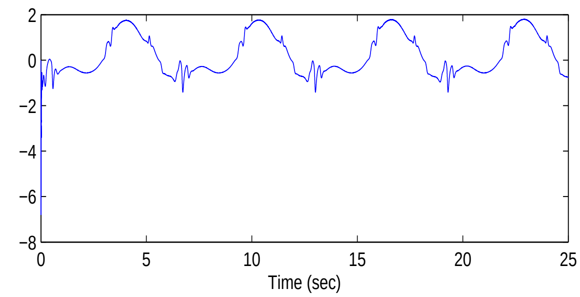

Fig. 4. Control input u_2

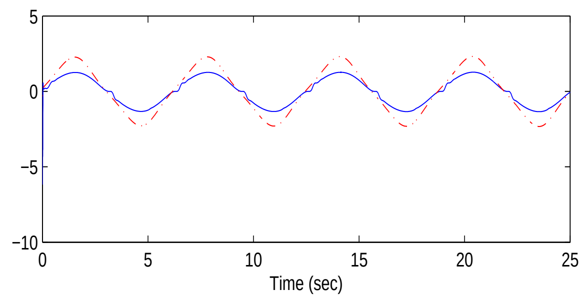

Fig. 5. Unknown function $\nu_{1,1}$ ("- -") and its estimate $\varpi_{1,1} W_{1,1}^T S_{1,1}(Z_{1,1})$ ("—")

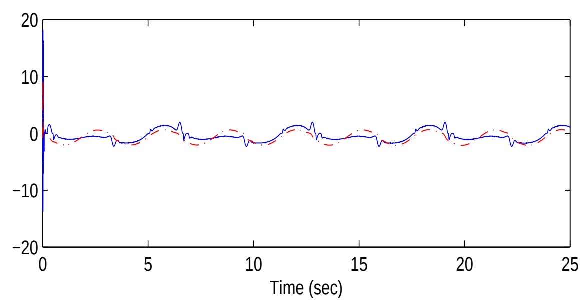

Fig. 6. Unknown function $\nu_{1,2}$ ("- -") and its estimate $\varpi_{1,2} W_{1,2}^T S_{1,2}(Z_{1,2})$ ("—")

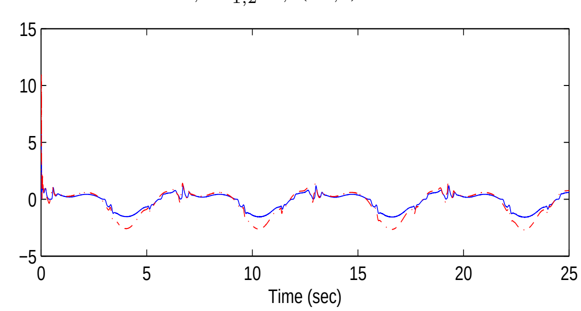

Fig. 7. Unknown function $\nu_{2,1}$ ("- -") and its estimate $\varpi_{2,1} W_{2,1}^T S_{2,1}(Z_{2,1})$ ("—")

Figs. 1–7 show the simulation results of applying controller

(80) and the NN weight updating laws (81) to system (78) for tracking reference signals $y_{rj}, j = 1, 2$ with $\beta = 0.001$ and the initial conditions $x = [0.5; 2; 1.3], x_r = [1.5; 0.8]$. According to [27], [28], there is a certain trade-off between the choice of the design parameters and the control action. The design parameters of the above controller are chosen as $c_{1,1} = 3, c_{1,2} = 6, c_{2,1} = 5, \Gamma_{1,1} = \Gamma_{1,2} = \Gamma_{2,1} = \mathrm{diag}\{2.0\}, \sigma_{1,1} = \sigma_{1,2} = \sigma_{2,1} = 1, \epsilon_{1,1} = \epsilon_{1,2} = \epsilon_{2,1} = 0.1$. Figs. 1–2 show the fairly good tracking performance. From Figs. 3–4, it follows that the control signals u_1 and u_2 are bounded and become periodic signals after 2s. Figs. 5–7 illustrate the learning ability of neural networks by plotting the nonlinear function as well as its estimate. Note that the tracking performance improves with increase of matching between the nonlinear function and its estimate. Hence, the proposed adaptive controller possesses the abilities of learning and controlling the unknown MIMO nonlinear system.

Example 2: Consider a SISO nonaffine pure-feedback system as in [2]

$$\begin{cases} \dot{x}_1 &= x_1 + x_2 + \frac{x_2^3}{5} \\ \dot{x}_2 &= x_1 x_2 + u + \frac{u^3}{7} + d(t) \\ y &= x_1, \end{cases} \tag{82}$$

where $d(t) = 0.1\cos(0.01t)\cos(y)$.

The control objective is to design a controller for system (82) such that the output y tracks a desired reference trajectories y_r, which is the output y_{r1} of the famous van der Pol oscillator (79), where $\beta = 0.2$ in this simulation. According to system (82), the adaptive NN controller is chosen according to (55) (i.e. $j = 1, \rho_1 = 2$) as follows:

$$u = -c_2 z_2 - \varpi_2 \hat{W}_2^T S_2(Z_2), \tag{83}$$

where $z_1 = x_1 - x_{r1}, z_2 = x_2 - \alpha_1, Z_1 = [x_1, x_{r2}]^T$ and $Z_2 = [x, \partial\alpha_1/\partial x_1, \zeta_1, \alpha_1]^T$ with $\alpha_1 = -c_1 z_1 - \varpi_1 \hat{W}_1^T S_1(Z_1)$, $\zeta_1 = \frac{\partial \alpha_1}{\partial x_{r1}}\dot{x}_{r1} + \frac{\partial \alpha_1}{\partial x_{r2}}\dot{x}_{r2} + \frac{\partial \alpha_1}{\partial \hat{W}_1}\dot{\hat{W}}_1$, $\varpi_i = \tanh\left(\frac{\hat{W}_i^T S_i(Z_i) z_i}{\epsilon_i}\right), i = 1, 2$ and NN weights $\hat{W}_i$ are updated by $\dot{\hat{W}}_i = \Gamma_i[S_i(Z_i)z_i - \sigma_i \hat{W}_i]$.

NN $\hat{W}_1^T S_1(Z_1)$ contains 9 nodes, with widths $\upsilon_{1,k} = 2$ $(k = 1, 2, \cdots, l_1)$ and centers $\mu_{1,k}$ $(k = 1, 2, \cdots, l_1)$ evenly spaced in $[-2.5, 2.5] \times [-7, 7]$; $\hat{W}_2^T S_2(Z_2)$ contains 243 nodes, with widths $\upsilon_{2,k} = 4$ $(k = 1, 2, \cdots, l_2)$ and centers $\mu_{2,k}$ $(k = 1, 2, \cdots, l_2)$ evenly spaced in $[-2.5, 2.5] \times [-3, 2] \times [-4, 0] \times [-6, 6] \times [-4, 5]$. The design parameters of the above controller are chosen as $c_1 = 9, c_2 = 3, \Gamma_1 = \Gamma_2 = \mathrm{diag}\{5.0\}, \sigma_1 = \sigma_2 = 0.1, \epsilon_1 = \epsilon_2 = 0.1$, and the initial conditions $x = [0.5; 1.8]$.

From Fig. 8, we can see that fairly good tracking performance is obtained. The boundedness of control signal u and NN weights $\varpi_1 W_1$ and $\varpi_2 W_2$ are shown in Figs. 9 and 10, respectively. Comparative tracking errors of the ISS-modular approach in [2] and the proposed approach in this paper are given in Fig. 11.

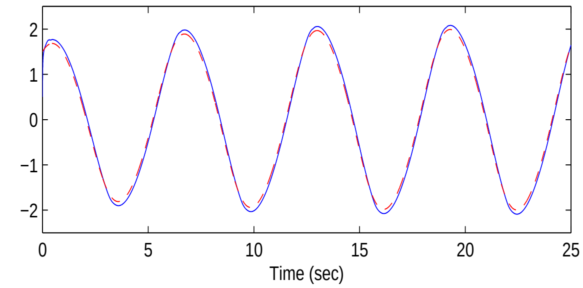

Fig. 8. Output y ("—") follows y_r ("- -")

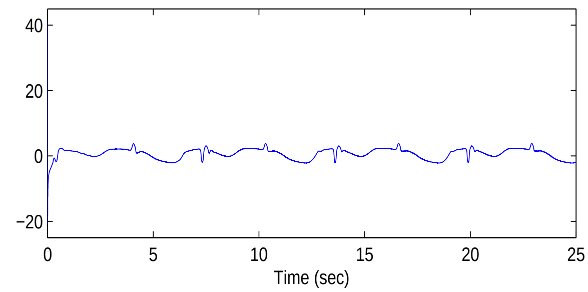

Fig. 9. Control input u

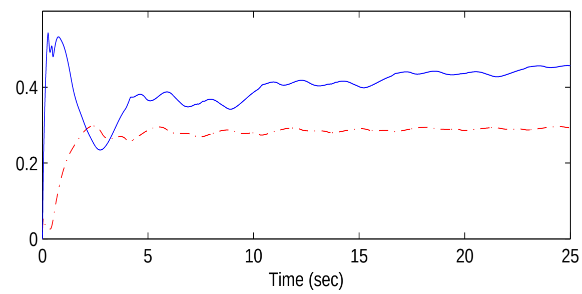

Fig. 10. L_2 norm of the NN weights: $\varpi_1 W_1$ ("—") and $\varpi_2 W_2$("- -")

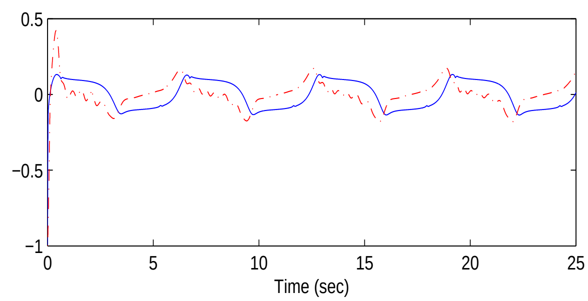

Fig. 11. Comparative tracking errors: the proposed approach ("—") in this paper and the ISS-modular approach ("- -") in [2]

Keeping all design parameters as before, Figs. 12–15 show the simulation results of applying controller (83) to system (82) for tracking reference signal $y_r = 0.5[\sin(t) + \sin(0.5t)]$, and confirm the effectiveness of the developed approach. Fig. 12 shows fairly good tracking performance obtained by the same adaptive controller (83). Figs. 13 and 14 show that control signal u and NN weights $\varpi_1 W_1$ and $\varpi_2 W_2$ are bounded, and Fig. 15 gives comparative tracking errors of the ISS-modular approach in [2] and the proposed approach in this paper. It is shown that the convergence of the ISS-modular approach in [2] is slower compared to the developed approach in this paper.

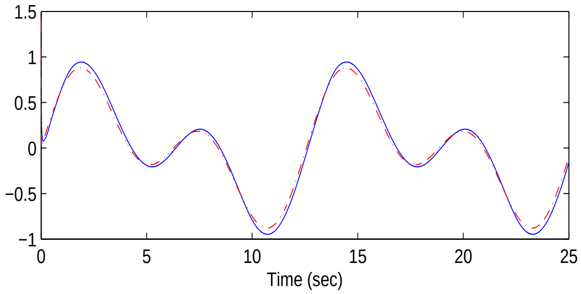

Fig. 12. Output y ("—") follows y_r ("- -")

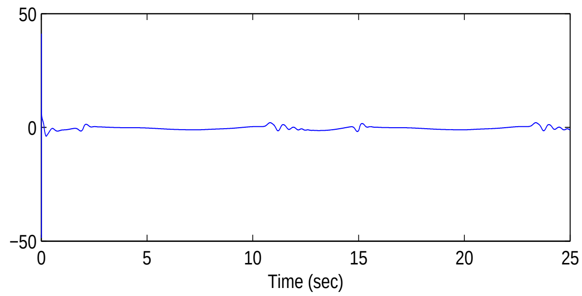

Fig. 13. Control input u

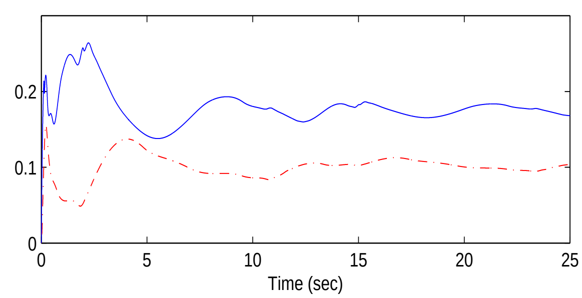

Fig. 14. L$_2$ norm of the NN weights: $\varpi_1 W_1$ ("—") and $\varpi_2 W_2$("- -")

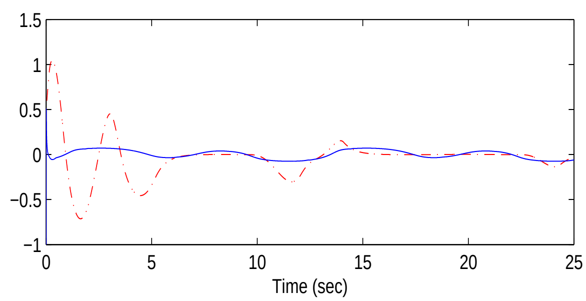

Fig. 15. Comparative tracking errors: the proposed NN control ("—") in this paper and the ISS-modular approach ("- -") in [2]

V. Conclusion

In this paper, we have proposed adaptive neural tracking control for a class of uncertain MIMO block-triangular nonaffine pure-feedback systems in the continuous-time form. Theoretical analysis and simulation studies suggest that our approach can tackle the difficulties in controlling MIMO block-triangular nonaffine pure-feedback systems and simplify the control design process. All signals in the closed-loop system are guaranteed to be semiglobal uniform ultimate bounded, and the system outputs are proven to converge to a small neighborhood of the desired trajectory. The adaptive NN scheme can be applied to a large number of uncertain MIMO pure-feedback nonlinear systems without repeating the complex controller design procedure.

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References

[1] S. S. Ge, C. C. Hang, T. H. Lee, and T. Zhang, *Stable Adaptive Neural Network Control*. Norwell, MA: Kluwer Academic, 2002.
[2] C. Wang, D. J. Hill, S. S. Ge, G. Chen, "An ISS-modular approach for adaptive neural control of pure-feedback systems," *Automatica*, vol. 42, no. 5, pp. 723-731, 2006.
[3] S. Jagannathan and P. He, "Neural-network-based state feedback control of a nonlinear discrete-time system in nonstrict feedback form," *IEEE Transactions on Neural Networks*, vol. 19, no. 12, pp. 2073-2087, 2008.
[4] Z. Chen and S. Jagannathan, "Generalized Hamilton-Jacobi-Bellman formulation-based neural network control of affine nonlinear discrete-time systems," *IEEE Transactions on Neural Networks*, vol. 19, no. 1, pp. 90-106, 2008.
[5] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Transactions on Automatic Control*, vol. 34, no. 11, pp. 1123-1131, 1989.
[6] K. S. Narendra and S. Mukhopadhyay, "Adaptive control of nonlinear multivariable system using neural networks," *Neural Networks*, vol. 7, no. 5, pp. 737-752, 1994.
[7] F. C. Chen and H. K. Khalil, "Adaptive control of a class of nonlinear discrete-time systems using neural networks," *IEEE Transactions on Automatic Control*, vol. 40, no. 5, pp. 791-801, 1995.
[8] S. S. Ge, "Robust adaptive NN feedback linearization control of nonlinea systems," *International Journal of Systems Science*, vol. 27, no. 12, pp. 1327-1338, 1996.
[9] S. S. Ge, C. C. Hang and T. Zhang, "Stable adaptive control for multivariable systems with a triangular control structure," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1221-1225, 2000.
[10] S. S. Ge, C. Wang, "Adaptive neural control of uncertain MIMO nonlinear systems," *IEEE Transactions on Neural Networks*, vol. 15, no. 3, pp. 674-692, 2004.
[11] M. Wang, C. Wang and S. S. Ge, "A novel ISS-Modular adaptive neural control of pure-feedback nonlinear systems," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 11, pp. 6559-6570, 2011.
[12] A. M. Zou, Z. G. Hou and M. Tan, "Adaptive control of a class of nonlinear pure-feedback systems using fuzzy backstepping approach," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 4, pp. 886-897, 2008.
[13] J. Na, X. Ren, C. Shang and Y. Guo, "Adaptive neural network predictive control for nonlinear pure feedback systems with input delay," *Journal of Process Control*, vol. 22, pp. 194-206, 2012.
[14] J. Na, X. Ren and D. Zheng, "Adaptive control for nonlinear pure-feedback systems with high-order sliding mode observer," *IEEE Transactions on Neural networks and Learning Systems*, vol. 24, no. 3, pp. 370-382, 2013.
[15] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.
[16] L. R. Hunt, and G. Meyer, "Stable inversion for nonlinear systems," *Automatica*, vol. 33, no. 8, pp. 1549-1554, 1997.
[17] A. Ferrara, and L. Giacomini, "Control of a class of mechanical systems with uncertainties via a constructive adaptive/second order VSC approach," *Journal of Dynamic Systems, Measurement and Control*, vol. 122, no. 1, pp. 33-39, 2000.
[18] B. Ren, S. S. Ge, C. Chen, C.-H. Fua and T. H. Lee, *Modeling, control and coordination of helicopter systems*, Springer-Verlag New York Inc, 2011.
[19] S. C. Malik, *Mathematical Analysis*, John Wiley & Son, New York, 1984.

[20] S. Haykin, *Neural networks: A comprehensive foundation*. (2nd ed.), New Jersey: Prentice-Hall, 1999.

[21] A. J. Kurdila, F. J. Narcowich, J. D. Ward, "Persistency of excitation in identification using neural networks," *SIAM Journal of Control and Optimization*, vol. 33, no.2, pp. 625-642, 1995.

[22] M. M. Polycarpou, P. A. Ioannou, "A robust adaptive nonlinear control design," *Automatica*, vol. 32, no. 3, pp. 423-427, 1996.

[23] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[24] T. Zhang, S. S. Ge and C. C. Hang,"Design and performance analysis of a direct adaptive controller of nonlinear systems," *Automatica*, vol. 35, no. 11, pp. 1809-1817, 1999.

[25] W. H. Young, "On the multiplication of successions of Fourier constants," *Proceedings of the Royal Society of London. Series A*, vol. 87, No. 596, pp. 331-339, 1912.

[26] Z. Qu, *Robust Control of Nonlinear Uncertain Systems*. New York: Wiley, 1998.

[27] F. Pozo, F. Ikhouane and J. Rodellar, "Numerical issues in backstepping control: Sensitivity and parameter tuning," *Journal of the Franklin Institute*, vol. 345, no. 8, pp. 891-905, 2008.

[28] Y. Chang, "Block backstepping control of MIMO systems," *IEEE Transactions on Automatic Control*, Vol. 56, No. 5, pp. 1191-1197, 2011.

[29] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1993.

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